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NOTE

ON A HARMONIOUS GRAPH CONJECTURE

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Let $K_n^{(2)}$ be the union of two complete graphs on n vertices which have precisely one vertex in common. Graham and Sloane have shown that $K_n^{(2)}$ is not harmonious for n odd, $K_4^{(2)}$ is harmonious, and $K_6^{(2)}$ is not harmonious. They also conjecture that $K_n^{(2)}$ is not harmonious except for $n=4$. Here, it is shown that if $K_n^{(2)}$ is harmonious, then n must be a sum of two squares.

1. Introduction

Let G be a finite connected graph with m edges, let Z_m be the integers modulo m, and let $\lambda(x)$ be a function from the vertex set of G to Z_m. $\lambda(x)$ is viewed as a labelling of the vertices x of G, and it induces an edge labelling by assigning the label $\mu(xy)=\lambda(x)+\lambda(y)$ to each edge xy of G (where addition is in Z_m). If G is not a tree, then G is called harmonious if there exists a labelling $\lambda(x)$ such that $\lambda(x)$ and $\mu(xy)$ are each one-to-one (which means that $\mu(xy)$ is bijective). Although we will not be concerned with trees, in this case $\lambda(x)$ is allowed to have exactly one duplicate.

The concept of a harmoniuous graph was introduced by Graham and Sloane [1] where they derive extensive results relating to this concept. Here, we focus on a particular one of their results. Namely, let $K_n^{(2)}$ ($n\geqslant 3$) be the graph consisting of two copies of K_n which have exactly one vertex in common (where Fig. 1 provides an illustration of $K_4^{(2)}$). It is shown in [1] that $K_n^{(2)}$ is not harmonious if n is odd; $K_4^{(2)}$ is harmonious (Fig. 1); and $K_6^{(2)}$ is not harmonious (which was determined by computer). The remaining cases were left unresolved, but Graham and Sloane make the conjecture that $K_n^{(2)}$ is harmonious only when $n=4$.

Here, we establish the following result.

Theorem. *If $K_n^{(2)}$ is harmonious, then n is the sum of two squares.*

As a consequence of this theorem we obtain that $K_6^{(2)}$ is not harmonious (without resorting to the computer). In fact, a more general corollary is that if $n\equiv 6 \pmod 8$, then $K_n^{(2)}$ is not harmonious. We also note that the set of integers

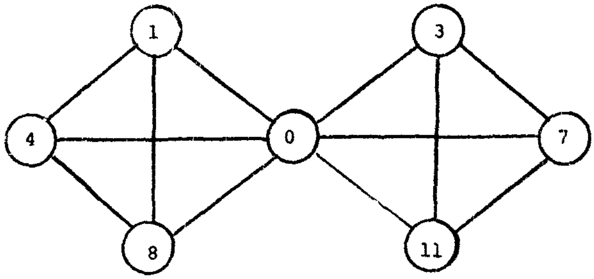

Fig 1 A harmonious labelling of $K_4^{(2)}$.

which can be represented as the sum of two squares has density zero. This follows from a result due to Landau [2] which says that the number of positive integers n not exceeding x which can be represented as the sum of two squares is asymptotic to $cx/\sqrt{\log x}$ for some positive constant c. From this, we can conclude that $K_n^{(2)}$ is not harmonious for almost all n.

2. Proof of the Theorem

Let K^L and K^R be two copies of K_n which make up $K_n^{(2)}$ (where L and R signify Left and Right) and let v be the single vertex which is common to K^L and K^R. A vertex in K^L will be called a left vertex and a vertex in K^R will be called a right vertex (so that v is both a left and a right vertex).

Now suppose $\lambda(x)$ is a labelling of $K_n^{(2)}$ which makes $K_n^{(2)}$ harmonious. Then $\lambda(x)$ is an integer mod $m = n(n-1)$ which is the number of edges in $K_n^{(2)}$. Since m is an even modulus, it is meaningful to speak of odd and even integers mod m. Then, let

s_L = number of vertices x in K^L with $\lambda(x)$ even,
t_L = number of vertices x in K^L with $\lambda(x)$ odd,
s_R = number of vertices x in K^R with $\lambda(x)$ even,
t_R = number of vertices x in K^R with $\lambda(x)$ odd.

We note that

$$s_L + t_L + s_R + t_R = 2n. \tag{1}$$

Now, for each edge xy in $K_n^{(2)}$, the edge label $\mu(xy)$ is an integer mod m and so also has a well-defined parity. We obtain the edges with even labels (and there are $\frac{1}{2}m$ such edges) by either adding two vertex labels in K^L with the same parity or adding two vertex labels in K^R with the same parity. Thus

$$\tfrac{1}{2}s_L(s_L-1)+\tfrac{1}{2}t_L(t_L-1)+\tfrac{1}{2}s_R(s_R-1)+\tfrac{1}{2}t_R(t_R-1)=\tfrac{1}{2}n(n-1) \tag{2}$$

since both sides represent the number of edges in $K_n^{(2)}$ which have an even label.

From (1) and (2) we obtain

$$s_L^2+t_L^2+s_R^2+t_R^2=n^2+n. \tag{3}$$

In a similar way, the edges with odd labels (which again are $\frac{1}{2}m$ in number) are obtained by adding vertex labels with opposite parity. Thus,

$$s_L t_L + s_R t_R = \tfrac{1}{2}n(n-1), \tag{4}$$

since the left side of (4) is the number of pairs of adjacent vertices in $K_n^{(2)}$ which have opposite parity. From (4), we have $2s_L t_L + 2s_R t_R = n(n-1)$ and upon subtracting from (3) we obtain

$$(s_L-t_L)^2+(s_R-t_R)^2=2n. \tag{5}$$

From (5), we see that $2n$ is a sum of two squares. But it is well known that if $2n$ is a sum of two squares (say $2n=a^2+b^2$), then n is also a sum of two squares (i.e., $n=\frac{1}{2}(a+b)^2+\frac{1}{2}(a-b)^2$) which is what we wanted to show.

3. Concluding remarks

The Graham–Sloane conjecture is that $K_n^{(2)}$ is harmonious only if $n=4$. In view of the above theorem (and the fact that $K_n^{(2)}$ is not harmonious for odd n), one might also be tempted to conjecture that $K_n^{(2)}$ is harmonious iff n is an even integer which is a sum of two squares. It would be interesting to settle the cases $n=8$ and $n=10$ in order to determine what is more plausible here.

References

[1] R L. Graham and N.J.A Sloane, On additive bases and harmonious graphs, SIAM J. Algebraic Discrete Methods 1 (4) (1980) 382–404.

[2] E. Landau, Ueber die Einteilung die positiven ganzen Zahlen in vier Klassen Nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate, Arch. Math. Phys (3) 13 (1908) 305–312.